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PHILOSOPHICAL TRANSACTIONS OF

IX. *Plane Stress and Plane Strain in Bipolar Co-ordinates.*

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§ 1. Introduction.

The problem of the equilibrium of an elastic solid under given applied forces is one of great difficulty and one which has attracted the attention of most of the great applied Mathematicians since the time of Euler. Unlike the kindred problems of hydrodynamics and electrostatics, it seems to be a branch of mathematical physics in which knowledge comes by the patient accumulation of special solutions rather than by the establishment of great general propositions. Nevertheless, the many and varied applications of this subject to practical affairs make it very desirable that these special solutions should be investigated, not only because of their intrinsic importance but also for the light which they often throw on the general problem. One of the most powerful methods of the mathematical physicist in the face of recalcitrant differential equations is to simplify his problem by reducing it to two dimensions. This simplification can only imperfectly be reproduced in the Nature of our three-dimensional world, but, in default of more general methods, it provides an invaluable weapon.

It was shown by Airy* that in the two-dimensional case the stresses may be derived by partial differentiations from a single stress function, and it was shown later† that, in the absence of body forces, this stress function satisfies the linear partial differential equation of the fourth order $\nabla^4\chi = 0$, where $\nabla^4 = \nabla^2 . \nabla^2$, and ∇^2 is the two-dimensional Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

It might have been expected that these results would have opened the way for a theory of two-dimensional elasticity of the same generality as the two-dimensional potential theory. This has not, however, been the case. This is due in part to the greater analytical difficulties which attend the discussion of the two-dimensional

* 'Brit. Assoc. Rep.,' 1862, p. 82.

† W. J. Ibbetson, 'Proc. Lond. Math. Soc.,' vol. xvii., 1886, p. 296. For a history of this part of the subject see Love's 'Elasticity,' 2nd edition, p. 17.

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solutions of $\nabla^4\chi = 0$ as compared with $\nabla^2\chi = 0$. The analogues of many of the important properties of the simpler equation have yet to be discovered if they exist at all. Some progress has been made, and in this connection we may mention the work of J. H. MICHELL* who established a general theory of inversion which, with some important differences, follows the potential theory fairly closely.

No doubt the analytical difficulties have been the chief obstacle to progress, but perhaps the theory has not in recent years received attention which it would have received but for a certain physical difficulty. A truly two-dimensional elastic system is not so easy of realisation as might seem to be the case at first sight. If the stresses are everywhere parallel to the xy plane and independent of z there will in general be a varying displacement parallel to z. If the displacements are everywhere parallel to the xy plane and independent of z this can only be secured by the application of a stress $\widehat{zz}$ which varies from point to point and is perpendicular to the xy plane. This difficulty was in a large measure removed by a theorem established by FILON, which has been called the theorem of generalised plane stress.† It states that if the average value of the stress $\widehat{zz}$ be taken throughout the thickness of a plate parallel to the xy plane, then the ordinary two-dimensional theory will give accurately the *average* stresses through the thickness of the plate if the elastic constants of the material are modified. If λ, μ denote the true elastic constants, λ must be replaced by $\lambda' = 2\lambda\mu/(\lambda+2\mu)$ while μ remains the same as before. This theorem attains an even greater importance when considered in the light of MICHELL's theorem,‡ that if a plate bounded by any number of bounding curves is in equilibrium under forces in its plane applied over the boundaries, then, provided the forces applied over each boundary taken separately are in equilibrium, the stresses are everywhere independent of the elastic constants. The hypothesis that the average value of $\widehat{zz}$ vanishes throughout the plate, while certainly not accurately true in the majority of cases, will probably give a very close approximation in the case of a *thin* plate where parallel faces are unstressed.

In the light of this generalisation it is of considerable importance that the two-dimensional problem should be worked out more thoroughly. The two-dimensional solutions of $\nabla^4\chi = 0$ have been investigated in several systems of curvilinear co-ordinates. Owing to the special importance of the problem of the rectangular beam the solutions in Cartesian co-ordinates have naturally received a considerable amount of attention. MICHELL gave the general form of the stress-function in polar co-ordinates, thus opening the way for the solution of the problem of a plate bounded by

* "The Inversion of Plane Stress," 'Proc. Lond. Math. Soc.,' 1901, vol. xxxiv., p. 134. Many of the results of the present paper can be obtained by an application of MICHELL's methods, but it has proved more convenient to proceed on different lines.

† 'Roy. Soc. Phil. Trans.,' A, 1903, vol. 201, pp. 63–155.

‡ 'Proc. Lond. Math. Soc.,' vol. xxi., 1900, p. 100.

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two concentric circles, or an infinite plate containing a circular hole under any given tractions applied over its boundaries. In his lectures at University College, London, in 1912, Prof. FILON gave the complete solution of this problem determining the stresses and displacements when the stresses on the boundaries are expanded in Fourier series, and I am not aware that this solution has ever been published. An outline of the solution in elliptic co-ordinates is given in LOVE's 'Elasticity.'*

In this paper the complete solution is given for bipolar co-ordinates, for which the co-ordinate curves are co-axial circles. This solution enables us to treat the problems of an infinite plate containing two circular holes, a semi-infinite plate bounded by a straight edge and containing one circular hole, and a circular disc with an eccentric circular hole.

In the second Section the equations are expressed in bipolar co-ordinates and formulæ are established for the displacements in terms of the stress-function.

In the third Section the stress-function is obtained in a convenient form and the terms giving rise to many valued displacements are separated out.

The fourth Section is devoted to the determination of the coefficients in the stress-function when the tractions over the boundaries are given in Fourier series, and to an examination of the convergence of the resulting series. From the results established in this section it appears that the solution is complete, for the stress-function can always be uniquely determined when the tractions are given, provided that the applied forces taken as a whole are in equilibrium.

The remaining sections are occupied with the examination of some of the simpler applications of the theory. Section 5 gives the solution for a circular disc with an eccentric hole (or a cylinder with eccentric bore) when the two boundaries are under different hydrostatic pressures. It is found that the solution of this problem can be expressed in finite terms. An important particular case of this problem is discussed in Section 6, namely, a semi-infinite plate with a straight, unstressed boundary and a circular hole under a uniform normal pressure. This will give the stresses near a rivet hole while the hot plastic rivet is being forced home under pressure. This solution is interesting from another point of view, for if the ratio of the radius of the hole to its distance from the edge is suitably adjusted, the point of greatest tension will be on the straight edge while the point of greatest stress difference is on the circular boundary. It thus suggests a crucial test for the rival theories of rupture,—the greatest tension theory and the greatest stress-difference theory.

Section 7 deals with a semi-infinite plate with an unstressed circular hole under tension parallel to its straight edge. The solutions are in the form of infinite series, but the more important aspects of the problem are illustrated by numerical tables.

* 2nd edition, p. 259.

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§ 2. The Co-ordinates.

Let us take curvilinear co-ordinates defined by the conjugate functions

$$\alpha+i\beta = \log\frac{x+i(y+a)}{x+i(y-a)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x, y are Cartesian co-ordinates and a is a positive real length. Solving for x, y, we have

$$x = \frac{a\sin\beta}{\cosh\alpha-\cos\beta}, \qquad y = \frac{a\sinh\alpha}{\cosh\alpha-\cos\beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Elements of arc measured along the normals to the curves α, β = *constant* are respectively $\delta\alpha/h$, $\delta\beta/h$, where

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2,$$

from which we have

$$h = (\cosh\alpha-\cos\beta)/a. \quad . \quad . \quad . \quad (3)$$

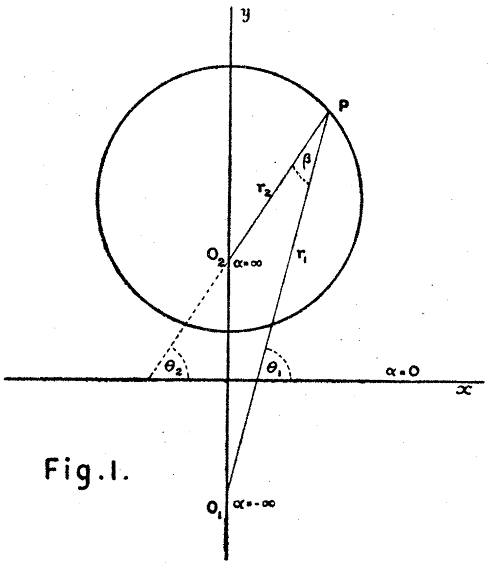

Fig. 1.

The general scheme of co-ordinates is shown in fig. 1. If O_1, O_2 are the points $0, -a$ and $0, a$ respectively and P any point in the plane, and if the radii from O_1, O_2 to P are of lengths r_1, r_2 and are inclined at angles θ_1, θ_2 to the axis of x, then $\alpha = \log r_1/r_2$ and $\beta = \theta_1-\theta_2$. The curves α = *constant* are a set of co-axial circles having O_1, O_2 for limiting points. The circles corresponding to positive values of α lie above the x-axis and those corresponding to negative values below, while the x-axis itself, which is the common radical axis, is given by $\alpha = 0$. The curves β = *constant* are circles, or rather arcs of circles passing through O_1, O_2 and cutting the first set of circles orthogonally. On the right-hand side of the y-axis β is positive and on the left-hand side negative, while on the y-axis $\beta = 0$, except on the segment O_1O_2, where $\beta = \pm\pi$. At infinity $\alpha = 0$, $\beta = 0$, and at O_1, O_2 we have $\alpha = -\infty$ and $+\infty$ respectively.

We have thus a set of co-ordinates adapted for the consideration of two-dimensional problems in which the region considered is—

(1) A finite region bounded internally by a circle and externally by a larger and non-concentric circle.

(2) A semi-infinite region bounded externally by a straight line and containing a circular hole.

(3) An infinite region containing two circular holes of any radii and centre distance.

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If the displacements in the directions normal to the curves α and β constant are u, v respectively, the strains are given by*

$$e_{\alpha\alpha} = h\frac{\partial u}{\partial \alpha} - v\frac{\partial h}{\partial \beta}, \qquad e_{\beta\beta} = h\frac{\partial v}{\partial \beta} - u\frac{\partial h}{\partial \alpha},$$

$$e_{\alpha\beta} = \frac{\partial}{\partial \alpha}(hv) + \frac{\partial}{\partial \beta}(hu),$$

and the corresponding components of stress by

$$\left.\begin{aligned} \widehat{\alpha\alpha} &= \lambda(e_{\alpha\alpha}+e_{\beta\beta}) + 2\mu e_{\alpha\alpha}, \\ \widehat{\beta\beta} &= \lambda(e_{\alpha\alpha}+e_{\beta\beta}) + 2\mu e_{\beta\beta}, \\ \widehat{\alpha\beta} &= \mu e_{\alpha\beta}. \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

These stresses may be derived from a stress-function, so that in rectangular co-ordinates

$$\widehat{xx} = \frac{\partial^2\chi}{\partial y^2}, \qquad \widehat{xy} = -\frac{\partial^2\chi}{\partial x\,\partial y}, \qquad \widehat{yy} = \frac{\partial^2\chi}{\partial x^2}.$$

Transforming these equations to curvilinear co-ordinates we obtain

$$\left.\begin{aligned} \widehat{\alpha\alpha} &= h\frac{\partial}{\partial\beta}\left(h\frac{\partial\chi}{\partial\beta}\right) - h\frac{\partial h}{\partial\alpha}\frac{\partial\chi}{\partial\alpha}, \\ \widehat{\beta\beta} &= h\frac{\partial}{\partial\alpha}\left(h\frac{\partial\chi}{\partial\alpha}\right) - h\frac{\partial h}{\partial\beta}\frac{\partial\chi}{\partial\beta}, \\ \widehat{\alpha\beta} &= -h\frac{\partial^2(h\chi)}{\partial\alpha\,\partial\beta} + h\chi\frac{\partial^2 h}{\partial\alpha\,\partial\beta}. \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

We will usually find it convenient to deal with $h\chi$ instead of χ itself, and in our particular co-ordinates these equations become

$$\left.\begin{aligned} a\widehat{\alpha\alpha} &= \left\{(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\beta^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cosh\alpha\right\}(h\chi), \\ a\widehat{\beta\beta} &= \left\{(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\alpha^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cosh\beta\right\}(h\chi), \\ a\widehat{\alpha\beta} &= -(\cosh\alpha - \cos\beta)\frac{\partial^2(h\chi)}{\partial\alpha\,\partial\beta}. \end{aligned}\right\} \quad . \quad . \quad (6)$$

We may note that

$$a(\widehat{\alpha\alpha} - \widehat{\beta\beta}) = (\cosh\alpha - \cos\beta)\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + 1\right)(h\chi), \quad . \quad . \quad . \quad (7)$$

* LOVE, 'Theory of Elasticity,' p. 54.

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so that if $h\chi$ and its second differential coefficients are finite at infinity ($\alpha = 0$, $\beta = 0$) we have there $\widehat{\alpha\alpha} = \widehat{\beta\beta} = h\chi$ and $\widehat{\alpha\beta} = 0$.

In the absence of body forces the stress-function satisfies $\nabla^4\chi = 0$. In curvilinear co-ordinates we have

$$\nabla^2 \equiv h^2\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right),$$

and, taking $h\chi$ as the dependent variable, we have in our co-ordinates

$$a\nabla^2\chi = \left\{(\cosh\alpha - \cos\beta)\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right) - 2\sinh\alpha\frac{\partial}{\partial\alpha} - 2\sin\beta\frac{\partial}{\partial\beta} + \cosh\alpha + \cos\beta\right\}(h\chi).$$

Repeating the operator, a little reduction leads us to the following transformation for $\nabla^4\chi = 0$:

$$\left(\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2\frac{\partial^2}{\partial\alpha^2} + 2\frac{\partial^2}{\partial\beta^2} + 1\right)(h\chi) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus by considering $h\chi$ instead of χ we have a linear equation with constant coefficients.

Before proceeding to the discussion of its solutions, we must investigate the method of determining the displacements corresponding to a given stress-function, in order that we may ascertain whether and under what conditions these are single-valued. This is particularly necessary in our case, as one of the co-ordinates, β, is itself many-valued.

Adding and subtracting the first two equations (4), and leaving the third as it stands, substituting for the stresses in terms of the stress-function, and for the strains in terms of the displacements, we obtain the following three equations:—

$$\frac{\partial}{\partial\alpha}\left\{\frac{\partial\chi}{\partial\alpha} - 2(\lambda+\mu)\frac{u}{h}\right\} + \frac{\partial}{\partial\beta}\left\{\frac{\partial\chi}{\partial\beta} - 2(\lambda+\mu)\frac{v}{h}\right\} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{\partial}{\partial\alpha}\left\{h^2\frac{\partial\chi}{\partial\alpha} + 2\mu hu\right\} - \frac{\partial}{\partial\beta}\left\{h^2\frac{\partial\chi}{\partial\beta} + 2\mu hv\right\} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{\partial}{\partial\alpha}\left\{h^2\frac{\partial\chi}{\partial\beta} + 2\mu hv\right\} + \frac{\partial}{\partial\beta}\left\{h^2\frac{\partial\chi}{\partial\alpha} + 2\mu hu\right\} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From the last two of these it appears that we may define a new function P such that

$$\frac{\partial P}{\partial\alpha} = h^2\frac{\partial\chi}{\partial\beta} + 2\mu hv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{\partial P}{\partial\beta} = h^2\frac{\partial\chi}{\partial\alpha} + 2\mu hu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\nabla^2 P = 0,$$

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and we have still to satisfy (9). Substituting for u, v in terms of P we have

$$h^2 \frac{\partial}{\partial \alpha}\left(\frac{1}{h^2}\frac{\partial P}{\partial \beta}\right) + h^2 \frac{\partial}{\partial \beta}\left(\frac{1}{h^2}\frac{\partial P}{\partial \alpha}\right) = \frac{\lambda+2\mu}{\lambda+\mu}\nabla^2\chi,$$

which may be re-arranged thus—

$$\frac{\partial}{\partial \alpha}\left\{\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial \chi}{\partial \alpha} - \frac{1}{h^2}\frac{\partial P}{\partial \beta}\right\} + \frac{\partial}{\partial \beta}\left\{\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial \chi}{\partial \beta} - \frac{1}{h^2}\frac{\partial P}{\partial \alpha}\right\} = 0.$$

It follows that a function Q exists such that

$$h^2 \frac{\partial Q}{\partial \alpha} = \frac{\partial P}{\partial \alpha} - \frac{\lambda+2\mu}{\lambda+\mu} h^2 \frac{\partial \chi}{\partial \beta}, \quad \ldots \ldots \quad (14)$$

$$h^2 \frac{\partial Q}{\partial \beta} = -\frac{\partial P}{\partial \beta} + \frac{\lambda+2\mu}{\lambda+\mu} h^2 \frac{\partial \chi}{\partial \alpha}. \quad \ldots \ldots \quad (15)$$

Eliminating P by differentiating with regard to β and α respectively, and adding. we have

$$\frac{\partial^2 (hQ)}{\partial \alpha\, \partial \beta} - Q\frac{\partial^2 h}{\partial \alpha\, \partial \beta} = \frac{\lambda+2\mu}{2(\lambda+\mu)}\left\{\frac{1}{h}\frac{\partial}{\partial \alpha}\left(h^2\frac{\partial \chi}{\partial \alpha}\right) - \frac{1}{h}\frac{\partial}{\partial \beta}\left(h^2\frac{\partial \chi}{\partial \beta}\right)\right\},$$

which becomes in our co-ordinates

$$\frac{\partial^2 (hQ)}{\partial \alpha\, \partial \beta} = \frac{\lambda+2\mu}{2(\lambda+\mu)}\left\{\frac{\partial^2 (h\chi)}{\partial \alpha^2} - \frac{\partial^2 (h\chi)}{\partial \beta^2} - h\chi\right\}. \quad \ldots \ldots \quad (16)$$

There is, however, a further condition to be satisfied by Q corresponding to the condition $\nabla^2 P = 0$. Differentiating (14) and (15) with regard to α, β respectively, and subtracting we have

$$\frac{\partial}{\partial \alpha}\left(h^2\frac{\partial Q}{\partial \alpha}\right) - \frac{\partial}{\partial \beta}\left(h^2\frac{\partial Q}{\partial \beta}\right) = -\frac{\lambda+2\mu}{\lambda+\mu}\left\{\frac{\partial}{\partial \alpha}\left(h^2\frac{\partial \chi}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(h^2\frac{\partial \chi}{\partial \alpha}\right)\right\},$$

or in our co-ordinates

$$\frac{\partial^2}{\partial \alpha^2}(hQ) - \frac{\partial^2}{\partial \beta^2}(hQ) - hQ = -\frac{2(\lambda+2\mu)}{\lambda+\mu}\frac{\partial^2 (hQ)}{\partial \alpha\, \partial \beta}. \quad \ldots \ldots \quad (17)$$

These two equations connecting Q and χ are consistent, for, if we eliminate Q by appropriate differential operators, we have

$$\left\{\frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2} - 1\right\}^2 (h\chi) = -4\frac{\partial^4 (h\chi)}{\partial \alpha^2\, \partial \beta^2},$$

which is readily seen to be identical with the condition $\nabla^4\chi = 0$, as given in (8). It is obvious that hQ satisfies the same differential equation, and hence it also is a solution of $\nabla^4 Q = 0$.

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We have therefore from (16)

$$hQ = \frac{\lambda+2\mu}{2(\lambda+\mu)}\iint\left\{\frac{\partial^2(h\chi)}{\partial\alpha^2}-\frac{\partial^2(h\chi)}{\partial\beta^2}-h\chi\right\}d\alpha\,d\beta, \qquad (18)$$

and from (12), (13) and (14), (15)

$$2\mu u = \frac{\mu}{\lambda+\mu}h\frac{\partial\chi}{\partial\alpha}-h\frac{\partial Q}{\partial\beta}, \qquad (19)$$

$$2\mu v = \frac{\mu}{\lambda+\mu}h\frac{\partial\chi}{\partial\beta}+h\frac{\partial Q}{\partial\alpha}. \qquad (20)$$

It is readily seen that these equations determine u and v apart possibly from rigid body displacements, for, although owing to the double integration an arbitrary function of α and an arbitrary function of β will appear in hQ, these will be determined by (17), except for functions of α or β, which make its left-hand side vanish identically. The only possible arbitrary terms in hQ are therefore given by $hQ = a\text{A}(\cosh\alpha+\cos\beta)+\text{B}(\cosh\alpha-\cos\beta)+\text{C}a\sinh\alpha+\text{D}a\sin\beta$, or

$$Q = \text{A}r^2+a\text{B}+\text{C}y+\text{D}x$$

where r is the distance from the origin. These give rise to terms in u, v corresponding to motions of pure translation and rigid body rotation about the origin.

§ 3. The Stress-Function.

Turning now to the consideration of the possible forms for the stress-function in these co-ordinates, we note that the differential equation (8) can readily be solved by the ordinary method, and that its general solution is

$$h\chi = e^{\alpha}\phi_1(\alpha+i\beta)+e^{-\alpha}\phi_2(\alpha+i\beta)+e^{\alpha}\phi_3(\alpha-i\beta)+e^{-\alpha}\phi_4(\alpha-i\beta).$$

If we seek a solution of the type $h\chi = f(\alpha)\cos n\beta$ or $f(\alpha)\sin n\beta$, (8) shows that the differential equation for $f(\alpha)$ is

$$\left(\frac{d^4}{d\alpha^4}-2(n^2+1)\frac{d^2}{d\alpha^2}+n^4-2n^2+1\right)f(\alpha) = 0,$$

the solution of which is

$$f(\alpha) = \text{A}_n\cosh(n+1)\alpha+\text{B}_n\cosh(n-1)\alpha+\text{C}_n\sinh(n+1)\alpha+\text{D}_n\sinh(n-1)\alpha,$$

unless $n = 0$ or 1. In the latter case we have

$$f(\alpha) = \text{A}_1\cosh 2\alpha+\text{B}_1+\text{C}_1\sinh 2\alpha+\text{D}_1\alpha,$$

and when $n = 0$

$$f(\alpha) = \text{A}_0\cosh\alpha+\text{B}_0\alpha\cosh\alpha+\text{C}_0\sinh\alpha+\text{D}_0\alpha\sinh\alpha.$$

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If we now seek solutions for which $h\chi$ is a multiple of sinh $n\alpha$ or cosh $n\alpha$, we find the following solutions which are not included above—

$$h\chi = (\mathrm{E}\cos\beta + \mathrm{F}\sin\beta + \mathrm{G}\cosh\alpha + \mathrm{H}\sinh\alpha)\,\beta.$$

Since any constant multiples of x, y and any constant may be added to χ without affecting the stresses, it follows from (2) that any multiples of

$$\sinh\alpha,\ \sin\beta \ \text{ or } \ \cosh\alpha - \cos\beta$$

may be added to $h\chi$. This allows us to take the coefficients of cosh α, sinh α and sin β as zero. We have then the following general expression for $h\chi$:—

$$\begin{aligned} h\chi = {} & (\mathrm{E}\cos\beta + \mathrm{F}\sin\beta + \mathrm{G}\cosh\alpha + \mathrm{H}\sinh\alpha)\,\beta \\ & + (\mathrm{B}_0\cosh\alpha + \mathrm{D}_0\sinh\alpha)\,\alpha \\ & + (\mathrm{A}_1\cosh 2\alpha + \mathrm{B}_1 + \mathrm{C}_1\sinh 2\alpha + \mathrm{D}_1\alpha)\cos\beta \\ & + (\mathrm{A}'_1\cosh 2\alpha + \mathrm{C}'_1\sinh 2\alpha + \mathrm{D}'_1\alpha)\sin\beta \\ & + \sum_{n=2}^{\infty}\left\{\begin{aligned} & [\mathrm{A}_n\cosh(n+1)\alpha + \mathrm{B}_n\cosh(n-1)\alpha + \mathrm{C}_n\sinh(n+1)\alpha \\ & \qquad + \mathrm{D}_n\sinh(n-1)\alpha]\cos n\beta \\ & + [\mathrm{A}'\cosh(n+1)\alpha + \mathrm{B}'_n\cosh(n-1)\alpha + \mathrm{C}'_n\sinh(n+1)\alpha \\ & \qquad + \mathrm{D}'_n\sinh(n-1)\alpha]\sin n\beta. \end{aligned}\right\} \end{aligned} \quad \ldots \quad (21)$$

We have now to determine whether the displacements corresponding to this stress-function are single valued or not. The function (hQ) is easily obtained by simple integration from (16), and the arbitrary functions thus appearing can be determined by the aid of (17). We have

$$\begin{aligned} -\frac{\lambda+\mu}{\lambda+2\mu}(h\mathrm{Q}) = {} & (\mathrm{E}\cos\beta + \mathrm{F}\sin\beta + \mathrm{G}\cosh\alpha + \mathrm{H}\sinh\alpha)\,\alpha \\ & - (\mathrm{B}_0\cosh\alpha + \mathrm{D}_0\sinh\alpha)\,\beta \\ & - (\mathrm{A}_1\sinh 2\alpha + \mathrm{C}_1\cosh 2\alpha + \mathrm{D}'_1\beta)\sin\beta \\ & + (\mathrm{A}'_1\sinh 2\alpha + \mathrm{C}'_1\cosh 2\alpha - \mathrm{D}_1\beta)\cos\beta \\ & + \sum_{n=2}^{\infty}\left\{\begin{aligned} & [\mathrm{A}'_n\sinh(n+1)\alpha + \mathrm{B}'_n\sinh(n-1)\alpha + \mathrm{C}'_n\cosh(n+1)\alpha \\ & \qquad + \mathrm{D}'_n\cosh(n-1)\alpha]\cos n\beta \\ & - [\mathrm{A}_n\sinh(n+1)\alpha + \mathrm{B}_n\sinh(n-1)\alpha + \mathrm{C}_n\cosh(n+1)\alpha \\ & \qquad + \mathrm{D}_n\cosh(n-1)\alpha]\sin n\beta. \end{aligned}\right\} \end{aligned} \quad \ldots \quad (22)$$

It is clear, from the general expressions for $h\chi$ and hQ, that the only terms which can possibly give rise to many-valued displacements are

$$\begin{aligned} h\chi = {} & (\mathrm{E}\cos\beta + \mathrm{F}\sin\beta + \mathrm{G}\cosh\alpha + \mathrm{H}\sinh\alpha)\,\beta \\ & + (\mathrm{B}_0\cosh\alpha + \mathrm{D}_0\sinh\alpha + \mathrm{D}_1\cos\beta + \mathrm{D}'_1\sin\beta)\,\alpha, \end{aligned}$$

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and the corresponding terms in $h\mathrm{Q}$

$$-\frac{\lambda+\mu}{\lambda+2\mu}h\mathrm{Q} = (\mathrm{E}\cos\beta+\mathrm{F}\sin\beta+\mathrm{G}\cosh\alpha+\mathrm{H}\sinh\alpha)\,\alpha$$
$$-(\mathrm{B}_0\cosh\alpha+\mathrm{D}_0\sinh\alpha+\mathrm{D}_1\cos\beta+\mathrm{D}'_1\sin\beta)\,\beta.$$

From (19) and (20) we may now find the corresponding displacements u, v. Each of these is found to contain a multiple of the many-valued co-ordinate β. Equating the coefficients of these terms to zero we have the following relations :—

$$\left.\begin{aligned} \mathrm{E}+\mathrm{G} &= 0, & \mathrm{B}_0+\mathrm{D}_1 &= 0,\\ \mu\mathrm{F}-(\lambda+2\mu)\,\mathrm{D}_0 &= 0, & \mu\mathrm{H}+(\lambda+2\mu)\,\mathrm{D}'_1 &= 0. \end{aligned}\right\} \quad (23)$$

We shall now show that these early terms correspond to the resultant of the forces and couple applied over the boundaries. For this purpose we shall require the following elementary forms of the stress-function :—

(1) For an isolated force X applied at the origin in the direction of the x-axis

$$\chi = -(2\pi)^{-1}\mathrm{X}\,(y\theta-\nu x\log r)$$

where r, θ as usual denote polar co-ordinates and $\nu = \mu/(\lambda+2\mu)$.

(2) For an isolated force Y applied at the origin in the direction of the y-axis

$$\chi = (2\pi)^{-1}\mathrm{Y}\,(x\theta+\nu y\log r).$$

(3) For a point couple of moment L applied at the origin in a positive sense

$$\chi = -(2\pi)^{-1}\mathrm{L}\theta.$$

(4) For a centre of pressure radiating uniformly from the origin

$$\chi = \log r,$$

Inserting the relations (23) necessary to ensure single-valued displacements our early terms become

$$h\chi = \mathrm{G}\,(\cosh\alpha-\cos\beta)\,\beta+\beta_0\,(\cosh\alpha-\cos\beta)\,\alpha$$
$$+\mathrm{F}\,(\beta\sin\beta+\nu\alpha\sinh\alpha)+\mathrm{H}\,(\beta\sinh\alpha-\nu\alpha\sin\beta)$$

or

$$\chi = a\mathrm{G}\beta+a\mathrm{B}_0\alpha+\mathrm{F}\,(x\beta+\nu y\alpha)+\mathrm{H}\,(y\beta-\nu x\alpha). \quad (24)$$

Now $a\mathrm{G}\beta = a\mathrm{G}\,(\theta_1-\theta_2)$ and hence this term represents a couple of moment $2\pi a\mathrm{G}$ applied at $\alpha = \infty$ and an equal and opposite couple applied at $\alpha = -\infty$. The term $a\beta_0\alpha$ represents two equal and opposite centres of radial pressure at these same points.

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We also have

$$\mathrm{H}\,(y\beta-\nu x\alpha) = \mathrm{H}\,\{(y+a)\,\theta_1-\nu x\log r_1\}-\mathrm{H}\{(y-a)\,\theta_2-\nu x\log r_2\}$$
$$-a\mathrm{H}\,(\theta_1+\theta_2).$$

This corresponds to a force $2\pi\mathrm{H}$ applied at $\alpha = +\infty$ parallel to the x-axis and an equal and opposite force applied at $\alpha = -\infty$ (thus forming a couple of moment $4\pi a\mathrm{H}$) and point couples each of moment $2\pi a\mathrm{H}$ applied at these same points (see fig. 2).

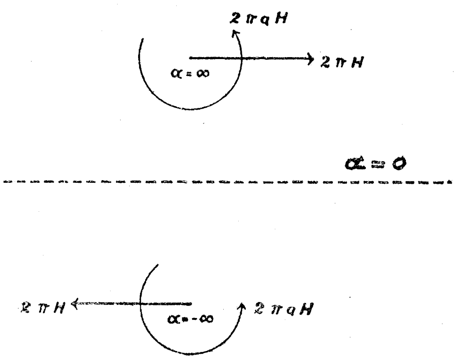

Fig. 2.

Finally

$$\mathrm{F}\,x\beta+\nu y\alpha) = \mathrm{F}\,\{x\theta_1+\nu\,(y+a)\log r_1\}$$
$$-\mathrm{F}\,\{x\theta_2+\nu\,(y-a)\log r_2\}$$
$$-a\nu\mathrm{F}\log r_1 r_2.$$

This corresponds to forces each equal to $2\pi\mathrm{F}$, acting at the points $\alpha = \pm\infty$ and each directed towards the origin, together with two equal like centres of uniform pressure at the same points. This brings to light a new solution corresponding to the last term.

Expressed in our co-ordinates we have

$$\log r_1 r_2 = 2\log(2a)-2\log(\cosh\alpha-\cos\beta),$$

and the corresponding form of $h\chi$ is, apart from constants,

$$h\chi = (\cosh\alpha-\cos\beta)\log(\cosh\alpha-\cos\beta).$$

It is easily seen that this can be expanded in a Fourier series which is included in our general expression for $h\chi$, but that the expansion is different on opposite sides of the line $\alpha = 0$. For this reason we shall find it convenient to include a term of this form whenever the region under consideration includes parts above and below the axis of x, *i.e.*, when it is bounded by two circles neither of which encloses the other.

It will be noted that, taken together, the early terms allow for the most general resultant forces acting over the two circular boundaries enclosing the two points $\alpha = +\infty$, $\alpha = -\infty$, subject to the condition that the forces acting over the two boundaries considered together form a system in equilibrium. If it is desired to investigate problems for which this condition is not satisfied we can readily obtain the necessary additional solutions. They will be

$$\left.\begin{aligned}\chi &= (y+a)\,\theta_1-\nu x\log r_1\\ \chi &= x\theta_1+\nu\,(y+a)\log r_1\\ \chi &= \theta_1\end{aligned}\right\} \qquad (25)$$

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corresponding to forces and couple applied at $\alpha = -\infty$, and similar terms in θ_2, $\log r_2$ corresponding to forces and couple applied at $\alpha = +\infty$. The corresponding forms of $h\chi$ can be expanded in series which are included in our general form, but here again the expansions are different on opposite sides of $\alpha = 0$ and diverge for $\alpha = 0$, $\beta = 0$ together, *i.e.*, at infinity. This divergence corresponds to the obvious fact that forces or couples must be applied at infinity to maintain equilibrium.

Owing to difficulties of this kind we shall find it convenient to insert the appropriate terms corresponding to the resultant force and couple over a boundary and to investigate the stress-function corresponding to the remaining applied forces which will be in statical equilibrium for each boundary.

Let us write for brevity

$$\left.\begin{aligned}\phi_n(\alpha) &= \mathrm{A}_n \cosh(n+1)\alpha + \mathrm{B}_n \cosh(n-1)\alpha + \mathrm{C}_n \sinh(n+1)\alpha + \mathrm{D}_n \sinh(n-1)\alpha \\ \psi_n(\alpha) &= \mathrm{A}'_n \cosh(n+1)\alpha + \mathrm{B}'_n \cosh(n-1)\alpha + \mathrm{C}'_n \sinh(n+1)\alpha + \mathrm{D}_n' \sinh(n-1)\alpha,\end{aligned}\right\} \quad (26)$$

if $n \geqq 2$ and

$$\left.\begin{aligned}\phi_1(\alpha) &= \mathrm{A}_1 \cosh 2\alpha + \mathrm{B}_1 + \mathrm{C}_1 \sinh 2\alpha \\ \psi_1(\alpha) &= \mathrm{A}'_1 \cosh 2\alpha + \mathrm{C}'_1 \sinh 2\alpha.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Setting aside the terms corresponding to the resultant forces and couples over the separate boundaries we have

$$\begin{aligned}h\chi &= \{\mathrm{B}_0\alpha + \mathrm{K}\log(\cosh\alpha - \cos\beta)\}(\cosh\alpha - \cos\beta) \\ &\quad + \sum_{n=1}^{\infty}\{\phi_n(\alpha)\cos n\beta + \psi_n(\alpha)\sin n\beta\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)\end{aligned}$$

where the term in K may be omitted when the region considered lies entirely on one side of the line

§ 4. Boundary Conditions.

Let us consider a plate bounded by two curves $\alpha = \alpha_1, \alpha_2$. We may suppose $\alpha_1 > \alpha_2$ and $\alpha_1 > 0$. Then, if $\alpha_2 > 0$ we have a finite plate bounded internally and externally by circles which are not concentric, if $\alpha_2 < 0$ we have an infinite plate containing two circular holes, and by suitably choosing the values of a, α_1, α_2 we can make the circular boundaries in either case of any desired radius and centre distance. In particular if $\alpha_2 = 0$, we have a semi-infinite plate bounded by a straight edge and containing a circular hole. Suppose that such a plate is in equilibrium under given normal and tangential forces applied over the boundaries $\alpha = \alpha_1, \alpha_2$, so that we are given over $\alpha = \alpha_1$,

$$\left.\begin{aligned}a\widehat{\alpha\beta} &= a_0 + \sum_1^{\infty}(a_n \cos n\beta + b_n \sin n\beta), \\ a\widehat{\alpha\alpha} &= c_0 + \sum_1^{\infty}(c_n \cos n\beta + d_n \sin n\beta),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

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while over $\alpha = \alpha_2$ we have similar expansions in which a_0, a_n, b_n, c_0, c_n, d_n are replaced by a'_0, a'_n, b'_n, c'_0, c'_n, d'_n.

If the tractions applied over the circle $\alpha = \alpha_1$ are statically equivalent to forces X, Y at its centre, and a couple of moment L, then

$$X = \int_0^{2\pi}\left\{\widehat{\alpha\alpha}\,\frac{\partial x}{\partial \alpha} - \widehat{\alpha\beta}\,\frac{\partial y}{\partial \alpha}\right\} d\beta,$$

$$Y = \int_0^{2\pi}\left\{\widehat{\alpha\alpha}\,\frac{\partial y}{\partial \alpha} + \widehat{\alpha\beta}\,\frac{\partial x}{\partial \alpha}\right\} d\beta,$$

$$L = -\frac{a^2}{\sinh \alpha_1}\int_0^{2\pi}\frac{\widehat{\alpha\beta}\,d\beta}{\cosh \alpha - \cos \beta}.$$

The coefficients of $\widehat{\alpha\alpha}$, $\widehat{\beta\beta}$ can readily be expanded in Fourier series. We have, in fact, since $\alpha_1 > 0$,

$$\frac{\partial x}{\partial \alpha} = -a\frac{\sinh \alpha \sin \beta}{(\cosh \alpha - \cos \beta)^2} = -2a\sum_1^\infty ne^{-n\alpha_1}\sin n\beta,$$

$$\frac{\partial y}{\partial \alpha} = -a\frac{(\cosh \alpha \cos \beta - 1)}{(\cosh \alpha - \cos \beta)^2} = -2a\sum_1^\infty ne^{-n\alpha_1}\cos n\beta,$$

and

$$\sinh \alpha_1 (\cosh \alpha_1 - \cos \beta)^{-1} = 1 + 2\sum_1^\infty e^{-n\alpha_1}\cos n\beta.$$

Substituting these and the expansions for $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ in the expressions for X, Y, L, and integrating, we have

$$X = 2\pi\sum_1^\infty n(a_n - d_n)e^{-n\alpha_1},$$

$$Y = -2\pi\sum_1^\infty n(b_n + c_n)e^{-n\alpha_1},$$

$$L = -2\pi a\,\mathrm{cosech}^2\alpha_1\sum_1^\infty a_n e^{-n\alpha_1}.$$

The corresponding components of the resultant of the forces applied over $\alpha = \alpha_2$ can be obtained in a similar way. We must, however, remember that in this case the forces act from that side of the boundary for which $\alpha < \alpha_2$, whereas in the case of the first boundary they acted from the side for which $\alpha < \alpha_1$. We obtain, if $\alpha_2 > 0$,

$$X' = -2\pi\sum_1^\infty n(a'_n - d'_n)e^{-n\alpha_2},$$

$$Y' = 2\pi\sum_1^\infty n(b'_n + c'_n)e^{-n\alpha_2},$$

$$L' = 2\pi a\,\mathrm{cosech}^2\alpha_2\sum_1^\infty a'_n e^{-n\alpha_2}.$$

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If $\alpha_2 < 0$ there are some differences of sign owing to the different Fourier expansions for the direction cosines. We have

$$X' = -2\pi \sum_1^\infty n\,(a'_n + d'_n)\, e^{n\alpha_2},$$

$$Y' = -2\pi \sum_1^\infty n\,(b'_n - c'_n)\, e^{n\alpha_2},$$

$$L' = -2\pi a \,\text{cosech}^2\alpha_2 \sum_1^\infty a'_n e^{n\alpha_2}.$$

Hence, if the forces acting on each boundary are statically in equilibrium, we have

$$\left.\begin{aligned} &\sum_1^\infty n\,(a_n - d_n)\, e^{-n\alpha_1} = 0, \qquad &&\sum_1^\infty n\,(b_n + c_n)\, e^{-n\alpha_1} = 0, \\ &\sum_1^\infty a_n e^{-n\alpha_1} = 0, &&\sum_1^\infty a'_n e^{\pm n\alpha_2} = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (30)$$

with, if $\alpha_2 > 0$,

$$\sum_1^\infty n\,(a'_n - d'_n)\, e^{-n\alpha_2} = 0, \qquad \sum_1^\infty n\,(b'_n + c'_n)\, e^{-n\alpha_2} = 0, \quad . \quad . \quad . \quad . \quad (31)$$

or, if $\alpha_2 < 0$,

$$\sum_1^\infty n\,(a'_n + d'_n)\, e^{n\alpha_2} = 0, \qquad \sum_1^\infty n\,(b'_v - c'_n)\, e^{n\alpha_2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

We will now show that it is possible to determine a stress-function of the form (28) which gives the appropriate stresses over $\alpha = \alpha_1, \alpha_2$, and which gives no stress at infinity if the region considered extends so far.

By the aid of (6) we can calculate the stresses corresponding to the stress-function (28). We obtain

$$2a\widehat{\alpha\alpha} = K\,(1 - 2\cosh^2\alpha) - 2B_0 \sinh\alpha \cosh\alpha + 2\phi_1(\alpha)$$
$$+ 2\,(K\cosh\alpha + B_0\sinh\alpha)\cos\beta - K\cos 2\beta$$
$$+ \sum_{n=1}^{\infty} \left\{ \begin{aligned} &[(n+1)(n+2)\,\phi_{n+1}(\alpha) - 2\cosh\alpha\,(n^2-1)\,\phi_n(\alpha) \\ &\qquad\qquad + (n-1)(n-2)\,\phi_{n-1}(\alpha)]\cos n\beta \\ &+ [(n+1)(n+2)\,\psi_{n+1}(\alpha) - 2\cosh\alpha\,(n^2-1)\,\psi_n(\alpha) \\ &\qquad\qquad + (n-1)(n-2)\,\psi_{n-1}(\alpha)]\sin n\beta \\ &- 2\sinh\alpha\,[\phi'_n(\alpha)\cos n\beta + \psi'_n(\alpha)\sin n\beta]. \end{aligned} \right\}$$

and

$$2a\widehat{\alpha\beta} = \psi'_1(\alpha) - 2\,(K\sinh\alpha + B_0\cosh\alpha)\sin\beta + B_0\sin 2\beta$$
$$+ \sum_{n=1}^{\infty} \left\{ \begin{aligned} &[(n+1)\,\psi'_{n+1}(\alpha) - 2\cosh\alpha n\psi'_n(\alpha) + (n-1)\,\psi'_{n-1}(\alpha)]\cos n\beta, \\ &- [(n+1)\,\phi'_{n+1}(\alpha) - 2\cosh\alpha n\phi'_n(\alpha) + (n-1)\,\phi'_{n-1}(\alpha)]\sin n\beta. \end{aligned} \right\}$$

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Identifying these with (29), we have the following relations from which to obtain the coefficients :—

$$\left.\begin{aligned}
&\phi_1(\alpha_1) = 2c_0+2\mathrm{B}_0 \sinh\alpha_1 \cos\alpha_1+2\mathrm{K}\cosh^2\alpha_1-\mathrm{K}\\
&1.2.3.\phi_2(\alpha_1) = 2c_1-2(\mathrm{K}\cosh\alpha_1+\mathrm{B}_0\sinh\alpha_1)+2\sinh\alpha_1\,\phi'_1(\alpha_1)\\
&2.3.4.\phi_3(\alpha_1)-2\cosh\alpha_1\,.\,1.2.3.\phi_2(\alpha_1) = 4c_2+2\mathrm{K}+4\sinh\alpha_1\,\phi'_2(\alpha_1)\\
&\quad\cdots\cdots\cdots\cdots\cdots\\
&n(n+1)(n+2)\phi_{n+1}(\alpha_1)-2\cosh\alpha_1(n-1)(n)(n+1)\phi_n(\alpha_1)+(n-2)(n-1)(n)\phi_{n-1}(\alpha_1)\\
&\qquad = 2nc_n+2n\sinh\alpha_1\,\phi'_n(\alpha_1) \qquad (n \geqq 3)
\end{aligned}\right\} \quad (33)$$

$$\left.\begin{aligned}
&n(n+1)(n+2)\psi_{n+1}(\alpha_1)-2\cosh\alpha_1(n-1)(n)(n+1)\psi_n(\alpha_1)+(n-2)(n-1)(n)\psi_{n-1}(\alpha_1)\\
&\qquad = 2nd_n+2n\sinh\alpha_1\,\psi'_n(\alpha_1) \qquad (n \geqq 1)
\end{aligned}\right\} \quad . \ . \ . \ (34)$$

$$\left.\begin{aligned}
&\psi'_1(\alpha_1) = 2a_0\\
&(n+1)\psi'_{n+1}(\alpha_1)-2\cosh\alpha_1\,n\psi'_n(\alpha_1)+(n-1)\psi'_{n-1}(\alpha_1) = 2a_n \qquad (n \geqq 1)
\end{aligned}\right\} \quad . \ (35)$$

$$\left.\begin{aligned}
&2\phi'_2(\alpha_1)-2\cosh\alpha_1\,\phi'_1(\alpha_1) = -2b_1-2(\mathrm{K}\sinh\alpha_1+\mathrm{B}_0\cosh\alpha_1)\\
&3\phi'_3(\alpha_1)-4\cosh\alpha_1\,\phi'_2(\alpha_1)+\phi'_1(\alpha_1) = -2b_2+\mathrm{B}_0\\
&\quad\cdots\cdots\cdots\cdots\cdots\\
&(n+1)\phi'_{n+1}(\alpha_1)-2\cosh\alpha_1\,n\phi'_n(\alpha_1)+(n-1)\phi'_{n-1}(\alpha_1) = -2b_n \qquad (n \geqq 3)
\end{aligned}\right\} \quad . \ (36)$$

Writing out equations (35), multiplying by $e^{-n\alpha_1}$ and adding, we have

$$(n+1)\psi'_{n+1}(\alpha_1)e^{-n\alpha_1}-n\psi'_n(\alpha_1)e^{-(n+1)\alpha_1} = 2\sum_{p=0}^{n} a_p e^{-p\alpha_1}$$

or

$$(n+1)\psi'_{n+1}(\alpha_1)-n\psi'_n(\alpha_1)e^{-\alpha_1} = 2e^{n\alpha_1}\sum_{p=0}^{n} a_p e^{-p\alpha_1}. \quad . \ . \ . \ (37)$$

Now, in virtue of (30), we may write the right-hand side of this

$$-2e^{n\alpha_1}\sum_{p=n+1}^{\infty} a_p e^{-p\alpha_1} = -2\sum_{r=1}^{\infty} a_{n+r}e^{-r\alpha_1},$$

and since $\Sigma a_n \cos n\beta$ is supposed convergent this tends to zero as n increases. Hence from (37) we see that the limit of $\psi'_{n+1}(\alpha_1)/\psi'_n(\alpha_1)$ as n increases is $e^{-\alpha_1}$, and hence the functions $\psi'_n(\alpha_1)$ are finite for all values of n and tend to zero as n increases, if the resultant couple acting on $\alpha = \alpha_1$ vanishes.

Multiplying (37) by $e^{n\alpha_1}$ and adding, we have

$$n\psi'_n(\alpha_1)e^{(n-1)\alpha_1} = 2\sum_{q=0}^{n-1}\sum_{p=0}^{q} a_p e^{(2q-p)\alpha_1} = 2\sum_{p=0}^{n-1}\sum_{q=p}^{n-1} a_p e^{(2q-p)\alpha_1},$$

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which, on effecting the summation with regard to q, leads at once to

$$n\psi'_n(\alpha_1) = 2 \operatorname{cosech} \alpha_1 \sum_{p=0}^{n-1} a_p \sinh(n-p)\alpha_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

for $n \geqq 1$.

Treating (34) in a similar way we have

$$n(n+1)(n+2)\psi_{n+1}(\alpha_1) - (n-1)(n)(n+1)\psi_n(\alpha_1)e^{-\alpha_1}$$
$$= 2e^{n\alpha_1} \sum_{p=1}^{n} p\left[d_p + \sinh \alpha_1 \psi'_p(\alpha_1)\right] e^{-p\alpha_1}. \quad . \quad . \quad . \quad (39)$$

We can readily show from (35) that

$$\sinh \alpha_1 \sum_{p=1}^{n} pe^{-p\alpha_1}\psi'_p(\alpha_1) = \tfrac{1}{2}n(n+1)\{\psi'_{n+1}(\alpha_1) - e^{-\alpha}\psi'_n(\alpha_1)\}e^{-n\alpha_1} - \sum_{p=1}^{n} pa_p e^{-p\alpha_1},$$

and hence (39) may be written

$$n(n+1)(n+2)\psi_{n+1}(\alpha_1) - (n-1)(n)(n+1)\psi_n(\alpha_1)e^{-\alpha_1} = n(n+1)\left[\psi'_{n+1}(\alpha_1) + e^{-\alpha_1}\psi'_n(\alpha_1)\right]$$
$$+ 2e^{n\alpha_1} \sum_{p=1}^{n} p(d_p - a_p)e^{-p\alpha_1}.$$

As in the case of $\psi'_n(\alpha_1)$, we can show that the right-hand side tends to zero as n increases if conditions (30) are fulfilled. Hence $\psi_n(\alpha_1)$ is finite for all values of n and tends to zero as n increases, and we have

$$(n-1)(n)(n+1)\psi_n(\alpha_1)e^{(n-1)\alpha_1} = 2\sum_{q=1}^{n-1}\sum_{p=1}^{q} p(d_p - a_p)e^{(2q-p)\alpha_1}$$
$$+ \sum_{q=1}^{n-1} q(q+1)e^{q\alpha_1}\left[\psi'_{q+1}(\alpha_1) - e^{-\alpha_1}\psi'_q(\alpha_1)\right],$$

which, on reduction, leads to

$$n(n^2-1)\psi_n(\alpha_1) = 2 \operatorname{cosech} \alpha_1 \sum_{p=0}^{n-1}\{(n-p)a_p \cosh(n-p)\alpha_1$$
$$+ (pd_p - a_p \coth \alpha_1)\sinh(n-p)\alpha_1\} \quad . \quad . \quad (40)$$

for $n \geqq 2$. Equations (34) do not determine $\psi_1(\alpha_1)$.

From (36) we have

$$2\phi'_2(\alpha_1)e^{-\alpha_1} - \phi'_1(\alpha_1)e^{-2\alpha_1} = \phi'_1(\alpha_1) - 2e^{-\alpha_1}(\mathrm{K}\sinh \alpha_1 + \mathrm{B}_0 \cosh \alpha_1) - 2b_1 e^{-\alpha_1}, \quad . \quad (41)$$

and if $n \geqq 2$

$$(n+1)\phi'_{n+1}(\alpha_1)e^{-n\alpha_1} - n\phi'_n(\alpha_1)e^{-(n+1)\alpha_1} = \phi'_1(\alpha_1) - 2\mathrm{K}e^{-\alpha_1}\sinh \alpha_1 - \mathrm{B}_0$$
$$-2\sum_{p=1}^{n} b_p e^{-p\alpha_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and hence, if the sequence $\phi_n(\alpha_1)$ is to converge for large values of n, we must have

$$\phi'_1(\alpha_1) = \mathrm{B}_0 + 2\mathrm{K}e^{-\alpha_1}\sinh \alpha_1 + 2\sum_{p=1}^{\infty} b_p e^{-p\alpha_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

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From (41) and (42) we have, if $n \geqq 2$,

$$n\phi'_n(\alpha_1)\, e^{(n-1)\alpha_1} = \phi'_1(\alpha_1) - B_0 + \sum_{q=1}^{n-1} (\phi'_1(\alpha_1) - 2Ke^{-\alpha_1} \sinh \alpha_1 - B_0)\, e^{2q\alpha_1}$$

$$-2 \sum_{q=1}^{n-1} \sum_{p=1}^{q} b_p e^{(2q-p)\alpha_1},$$

from which we obtain

$$n \sinh \alpha_1\, \phi'_n(\alpha_1) = (\phi'_1(\alpha_1) - B_0) \sinh n\alpha_1 - 2K \sinh (n-1)\alpha_1 \sinh \alpha_1$$
$$-2 \sum_{v=1}^{n-1} b_p \sinh (n-p)\alpha_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

for $n \geqq 2$, while $\phi'_1(\alpha_1)$ is given by (43).

Finally we have from (33), omitting the first equation of the series, if $n \geqq 2$,

$$n(n+1)(n+2)\, e^{-n\alpha_1}\phi_{n+1}(\alpha_1) - (n-1)(n)(n+1)\, e^{-(n+1)\alpha_1}\phi_n(\alpha_1)$$
$$= -2(B_0 + K)\, e^{-\alpha_1} \sinh \alpha_1$$
$$+2 \sum_{p=1}^{n} p\,(c_p + \sinh \alpha_1\, \phi'_p(\alpha_1))\, e^{-p\alpha_1}.$$

By the aid of (36) we can reduce the right-hand side to

$$n(n+1)(\phi'_{n+1}(\alpha_1) - e^{-\alpha_1}\phi'_n(\alpha_1))\, e^{-n\alpha_1} + 2 \sum_{p=1}^{n} p\,(c_p + b_p)\, e^{-p\alpha_1},$$

from which it appears that $\phi_{n+1}(\alpha_1)$ is finite and tends to zero as n increases, provided that the resultant of the applied forces over $\alpha = \alpha_1$ is zero. We then obtain for $n \geqq 2$,

$$n(n^2-1) \sinh \alpha_1\, \phi_n(\alpha_1) = (\phi'_1(\alpha_1) - B_0)\{n \cosh n\alpha_1 - \coth \alpha_1 \sinh n\alpha_1\}$$
$$-K\{(n-1) \sinh n\alpha_1 - (n+1) \sinh (n-2)\alpha_1\}$$
$$+2 \sum_{p=1}^{n-1} \{(pc_p + b_p \coth \alpha_1) \sinh (n-p)\alpha_1$$
$$-(n-p)\, b_p \cosh (n-p)\alpha_1\}, \quad . \quad . \quad . \quad . \quad (45)$$

while

$$2\phi_1(\alpha_1) = 2c_0 + B_0 \sinh 2\alpha_1 + K(2\cosh^2\alpha - 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

It appears that equations (38), (40), (43), (44), (45) and (46) give the values of $\phi_n(\alpha_1)$, $\psi_n(\alpha_1)$, $\phi'_n(\alpha_1)$, $\psi'_n(\alpha_1)$ for $n \geqq 1$ in terms of B_0, K and the given coefficients a_n, &c., with the exception of $\psi_1(\alpha_1)$.

Now we have only assumed $\alpha_1 > 0$ in order to establish the convergence of these functions, and hence the corresponding functions of α_2 will be given by the same formulæ with a'_n, b'_n, c'_n, d'_n substituted for a_n, b_n, c_n, d_n, provided that the conditions for convergence are satisfied. It may be shown that the new conditions of convergence are identical with (30) and (31), or (30) and (32), according as $\alpha_2 >$ or < 0.

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The formula for $\phi_1(\alpha_1)$ given in (43), which is itself a condition of convergence, will, however, be replaced by

$$\phi'_1(\alpha_2) = \mathrm{B}_0 + 2\mathrm{K}e^{\alpha_2} \sinh \alpha_2 + 2 \sum_{n=1}^{\infty} b'_n e^{n\alpha_2}, \quad \ldots \quad (47)$$

if $\alpha_2 < 0$.

From (26) we see that the coefficients A_n, B_n, C_n, D_n for $n \geqq 2$ are determined from $\phi_n(\alpha_1)$, $\phi_n(\alpha_2)$, $\phi'_n(\alpha_1)$, $\phi'_n(\alpha_2)$, and similarly A'_n, B'_n, C'_n, D'_n are determined from $\psi_n(\alpha_1)$, $\psi_n(\alpha_2)$, $\psi'_n(\alpha_1)$, $\psi'_n(\alpha_2)$,

The values of $\phi_1(\alpha_1)$, $\phi_1(\alpha_2)$, $\phi'_1(\alpha_1)$, $\phi'_1(\alpha_2)$ will give four equations to determine the three constants A_1, B_1, C_1, and the condition that they shall be consistent gives one relation between B_0 and K. The values of $\psi'_1(\alpha_1)$, $\psi'_1(\alpha_2)$ determine the two constants A'_1, C'_1, and $\psi_1(\alpha_1)$, $\psi_1(\alpha_2)$ are not otherwise determined.

We have thus just sufficient equations to determine the coefficients in (28) with the exception of B_0, K, between which we have found one relation. If $\alpha_2 > 0$, so that the region considered lies entirely on one side of the axis $\alpha = 0$, we may take $\mathrm{K} = 0$. If on the other hand $\alpha_2 < 0$ the condition that the stress shall vanish at infinity, which is $h\chi \to 0$ when α, $\beta \to 0$, gives one more relation between the coefficients, so that in either case B_0, K are determined.

We may therefore adopt the following method:—Insert terms of the type (24) or (25) corresponding to the resultant force and couple on each boundary, and calculate the residual stresses over the boundaries. These will now form systems in statical equilibrium over each boundary, and we have shown how to determine an appropriate function of the form (28).

The problem of finding the appropriate stress-function for given tractions over the boundaries might have been approached by investigating the values of $h\chi$ and its normal gradient on the boundaries, on the lines developed by MICHELL.* The direct method which we have adopted is, however, in most cases simpler in our particular co-ordinates.

There is an exception to this rule, namely, when a boundary is free from stress. In this case the boundary conditions assume a very simple form. From (6) we have

$$\frac{\partial}{\partial \alpha}(h\chi) = \text{const} = \rho, \text{ say} \quad \ldots \quad (48)$$

and

$$(\cosh \alpha - \cos \beta)\frac{\partial^2}{\partial \beta^2}(h\chi) - \sin \beta \frac{\partial}{\partial \beta}(h\chi) + \cosh \alpha\,(h\chi) = \rho \sinh \alpha$$

the solution of which is readily found to give

$$h\chi = \rho \tanh \alpha + \sigma(\cosh \alpha \cos \beta - 1) + \tau \sin \beta \quad \ldots \quad (49)$$

on the boundary considered.

* 'Proc. London Mathematical Society,' vol. xxi., 1900, p. 100.

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The relations (48) and (49) are the necessary and sufficient conditions that a boundary $\alpha =$ constant should be free from stress. The constants ρ, σ, τ are MICHELL's three constants of the boundary.

§ 5. A CYLINDER OR PIPE WITH ECCENTRIC BORE.

In this section we will consider the problem of a cylinder, whose cross-section is bounded by two non-concentric circles, which is subject to a uniform normal pressure over its internal surface and a different uniform normal pressure over its external surface. By FILON's theorem of generalised plane stress precisely the same analysis will give the average stresses in a plate of the same section under the same applied forces.

Let the boundaries of the cross-section be defined by $\alpha = \alpha_1$ for the internal boundary and $\alpha = \alpha_2$ for the external boundary. Then α_1, α_2 are positive and $\alpha_1 > \alpha_2$. Let the applied pressures be P_1, P_2 respectively, so that $\widehat{\alpha\alpha} = -P_1$ on $\alpha = \alpha_1$, $\widehat{\alpha\alpha} = -P_2$ on $\alpha = \alpha_2$ and $\widehat{\alpha\beta} = 0$ on both boundaries.

Let us assume

$$h\chi = B_0\alpha\,(\cosh\alpha - \cos\beta) + (A_1\cosh 2\alpha + B_1 + C_1\sinh 2\alpha)\cos\beta.$$

Calculating $\widehat{\alpha\alpha}$, $\widehat{\beta\beta}$, by means of (6) and applying the boundary conditions, we find the following values for the constants:—

$$B_0 = 2a\,M\,(P_1 - P_2)\cosh(\alpha_1 - \alpha_2)$$
$$A_1 = -aM\,(P_1 - P_2)\sinh(\alpha_1 + \alpha_2)$$
$$C_1 = aM\,(P_1 - P_2)\cosh(\alpha_1 + \alpha_2)$$
$$B_1 = aM\,\{P_1\cosh(\alpha_1 - \alpha_2)\sinh 2\alpha_2 - P_2\cosh(\alpha_1 - \alpha_2)\sinh 2\alpha_1 + (P_1 + P_2)\sinh(\alpha_1 - \alpha_2)\}$$

where, for brevity, we have written

$$M = \tfrac{1}{2}\operatorname{cosech}(\alpha_1 - \alpha_2)\,\{\sinh^2\alpha_1 + \sinh^2\alpha_2\}^{-1}$$

The most important aspect of the problem is the value of the stress $\widehat{\beta\beta}$ in the boundaries, for it is upon this that the strength of the cylinder will depend. This is most readily determined by (7), and we find without difficulty

$$\widehat{\alpha\alpha} - \widehat{\beta\beta} = 4M\,(P_1 - P_2)(\cosh\alpha - \cos\beta)\,\{\sinh(\alpha_1 + \alpha_2 - 2\alpha)\cos\beta - \sinh\alpha\cosh(\alpha_1 - \alpha_2)\}$$

so that on $\alpha = \alpha_1$

$$\widehat{\beta\beta} = -P_1 + 4\,(P_1 - P_2)\,M\,(\cosh\alpha_1 - \cos\beta)\{\sinh(\alpha_1 - \alpha_2)\cos\beta + \sinh\alpha_1\cosh(\alpha_1 - \alpha_2)\} \quad (50)$$

and on $\alpha = \alpha_2$

$$\widehat{\beta\beta} = -P_2 - 4\,(P_1 - P_2)\,M\,(\cosh\alpha_2 - \cos\beta)\{\sinh(\alpha_1 - \alpha_2)\cos\beta - \sinh\alpha_2\cosh(\alpha_1 - \alpha_1)\} \quad (51)$$

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In order to investigate these results further we will consider separately the cases when the cylinder is subject to *either* internal *or* external pressures. There is no greater difficulty in the consideration of the general case, should the necessity arise, except that the formulæ are correspondingly longer.

A Cylinder under Internal Pressure.

If we put $P_2 = 0$, we have on the external surface

$$\widehat{\beta\beta} = -4P_1 M (\cosh \alpha_2 - \cos \beta)\{\sinh (\alpha_1 - \alpha_2) \cos \beta - \sinh \alpha_2 \cosh (\alpha_1 - \alpha_2)\},$$

if d_1, d_2 denote the distances of the circles α_1, α_2 from the origin, r_1, r_2 their radii and d the distance apart of their centres, so that $d = d_2 - d_1$ we may show from (2) that

$$d_1 = a \coth \alpha_1, \qquad d_2 = a \coth \alpha_2$$

$$r_1 = a \operatorname{cosech} \alpha_1, \qquad r_2 = a \operatorname{cosech} \alpha_2$$

and

$$d_1 = (r_2^2 - r_1^2 - d^2)/2d, \qquad d_2 = (r_2^2 - r_1^2 + d^2)/2d$$

$$a^2 = \{r_2^2 - (r_1 + d)^2\} \{r_2^2 - (r_1 - d)^2\}/4d^2.$$

By means of these relations we can reduce the expression for $\widehat{\beta\beta}$ to the form

$$\widehat{\beta\beta} = \frac{2P_1 r_1^2 \{r_2^2 (r_2 - 2d \cos \beta)^2 - (r_1^2 - d^2)^2\}}{(r_1^2 + r_2^2)\{r_2^2 - (r_1 + d)^2\}\{r_2^2 - (r_1 - d)^2\}}.$$

From this and the obvious inequality $d < r_2 - r_1$ we easily see that—

(1) The numerically greatest stress is when $\beta = \pi$, *i.e.*, on the line of centres at the thinnest part of the cylinder. This is always a tension if P_1 is positive and is given by

$$\frac{2P_1 r_1^2 (r_2^2 + r_1^2 + 2r_2 d - d^2)}{(r_1^2 + r_2^2)(r_2^2 - r_1^2 - 2r_2 d + d^2)} \quad \ldots \ldots \ldots \quad (52)$$

(2) If the centre distance is greater than half the external radius there is minimum stress at the points corresponding to $\cos \beta = r_2/2d$. This is always negative when P_1 is positive and we have maximum compressions equal to

$$\frac{2P_1 r_1^2 (r_1^2 - d^2)^2}{(r_1^2 + r_2^2)\{r_2^2 - (r_1 + d)^2\}\{r_2^2 - (r_1 - d)^2\}}. \quad \ldots \ldots \quad (53)$$

This is always numerically less than the maximum tension. There is a secondary maximum at $\beta = 0$, *i.e.*, on the line of centres at the thickest part of the cylinder, which is equal to

$$\frac{2P_1 (r_2^2 + r_1^2 - 2r_2 d - d^2)}{(r_1^2 + r_2^2)(r_2^2 - r_1^2 + 2r_2 d + d^2)} \quad \ldots \ldots \ldots \quad (54)$$

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(3) If the centre distance is less than half the external radius, we have, in addition to the maximum tension (52), a minimum at $\beta = 0$ given by (54). There are no other maxima or minima and the stress decreases steadily from its value at the thinnest part of the cylinder to its value at the thickest part.

On the internal surface we have

$$\widehat{\beta\beta} = -\mathrm{P}_1 + 4\mathrm{P}_1\mathrm{M}\,(\cosh\alpha_1 - \cos\beta)\,\{\sinh(\alpha_1-\alpha_2)\cos\beta + \sinh\alpha_1\cosh(\alpha_1-\alpha_2)\},$$

or, expressed in terms of the radii and centre distance,

$$\widehat{\beta\beta} = -\mathrm{P}_1 + \frac{2\mathrm{P}_1 r_2^{\,2}\{(r_2^{\,2}-d^2)^2 - r_1^{\,2}(r_1+2d\cos\beta)^2\}}{(r_1^{\,2}+r_2^{\,2})\{r_2^{\,2}-(r_1-d)^2\}\{r_2^{\,2}-(r_1+d)^2\}}. \quad . \quad . \quad . \quad . \quad (55)$$

Hence it may be shown that

(1) If the centre distance is greater than one-half the internal radius the maximum stress in the internal surface occurs at the points corresponding to $\cos\beta = -r_1/2d$ and is

$$-\mathrm{P}_1 + \frac{2\mathrm{P}_1 r_2^{\,2}(r_2^{\,2}-d^2)^2}{(r_1^{\,2}+r_2^{\,2})\{r_2^{\,2}-(r_1-d)^2\}\{r_2^{\,2}-(r_1+d)^2\}} \quad . \quad . \quad . \quad . \quad . \quad (56)$$

(2) If the centre distance is less than one-half the internal radius the maximum stress is at $\beta = \pi$, *i.e.*, on the line of centres at the thinnest part of the cylinder. It is

$$-\mathrm{P}_1 + \frac{2\mathrm{P}_1 r_2^{\,2}(r_2^{\,2}+r_1^{\,2}-2r_1 d-d^2)}{(r_1^{\,2}+r_2^{\,2})(r_2^{\,2}-r_1^{\,2}-2r_1 d-d^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

(3) The minimum stress is at $\beta = 0$, the point where the line of centres meets the internal boundary at the thickest part of the cylinder. It is

$$-\mathrm{P}_1 + \frac{2\mathrm{P}_1 r_2^{\,2}(r_2^{\,2}+r_1^{\,2}+2r_1 d-d^2)}{(r_1^{\,2}+r_2^{\,2})(r_2^{\,2}-r_1^{\,2}+2r_1 d-d^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

This may be shown to be essentially positive if P is positive so that, as would be expected, the internal boundary is everywhere in a state of tension.

A Cylinder under External Pressure.

Putting $\mathrm{P}_1 = 0$ in (50) and (51) we have on the internal surface

$$\begin{aligned}\widehat{\beta\beta} &= -4\mathrm{P}_2\mathrm{M}\,(\cosh\alpha_1-\cos\beta)\,\{\sinh(\alpha_1-\alpha_2)\cos\beta+\sinh\alpha_1\cosh(\alpha_1-\alpha_2)\}\\ &= -\frac{2\mathrm{P}_2 r_2^{\,2}\{(r_2^{\,2}-d^2)^2-r_1^{\,2}(r_1+2d\cos\beta)^2\}}{(r_1^{\,2}+r_2^{\,2})\{r_2^{\,2}-(r_1-d)^2\}\{r_2^{\,2}-(r_1+d)^2\}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)\end{aligned}$$

and on the external surface

$$\begin{aligned}\widehat{\beta\beta} &= -\mathrm{P}_2+4\mathrm{P}_2\mathrm{M}\,(\cosh\alpha_2-\cos\beta)\,\{\sinh(\alpha_1-\alpha_2)\cos\beta-\sinh\alpha_2\cosh(\alpha_1-\alpha_2)\}\\ &= -\mathrm{P}_2 - \frac{2\mathrm{P}_2 r_1^{\,2}\{r_2^{\,2}(r_2-2d\cos\beta)^2-(r_1^{\,2}-d^2)^2\}}{(r_1^{\,2}+r_2^{\,2})\{r_2^{\,2}-(r_1-d)^2\}\{r^2-(r_1+d)^2\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)\end{aligned}$$

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Hence if the centre distance is less than half the internal radius the compression in the inner surface decreases steadily from a maximum at the thinnest part of the cylinder to a minimum at the thickest part; otherwise there is a minimum at each of the points and maxima at the points corresponding to $\cos\beta = -r_1/2d$. Similarly if the centre distance is less than half the external radius the compression in the outer surface decreases steadily from a maximum at the thinnest part of the cylinder to a minimum at the thickest part; if the centre distance exceeds this value the compression is a maximum at each of these points and minima at the points corresponding to $\cos\beta = r_1/2d$.

If in these results we put $d = 0$, we have, for a concentric tube under internal pressure, tensions at the inner and outer surfaces which are respectively

$$\frac{r_2^2+r_1^2}{r_2^2-r_1^2}P_1, \qquad \frac{2r_1^2}{r_2^2-r_1^2}P_1,$$

while for a tube under external pressure the compressions at the inner and outer surfaces are respectively

$$\frac{2r_2^2}{r_2^2-r_1^2}P_2, \qquad \frac{r_2^2+r_1^2}{r_2^2-r_1^2}P_2,$$

These are the well-known formulæ for thick tubes.

§ 6. A Semi-infinite Plate with a Circular Hole Subject to a Uniform Normal Pressure.

If in the results of the last section we put $\alpha_2 = 0$ and $P_2 = 0$, we have the solution for a semi-infinite plate containing a circular hole, which is subject to a uniform normal pressure, and bounded by a straight edge which is free from stress.

We have on the boundary of the hole

$$\widehat{\beta\beta} = -P_1+2P_1\operatorname{cosech}^2\alpha_1(\cosh^2\alpha_1-\cos^2\beta)$$

and on the straight edge

$$\widehat{\beta\beta} = -2P_1\operatorname{cosech}^2\alpha_1(1-\cos\beta)\cos\beta.$$

If r is the radius of the hole, d the perpendicular distance of its centre from the straight edge, and x the distance measured along the straight edge from the foot of the perpendicular,

$$d = a\coth\alpha, \qquad r = a\operatorname{cosech}\alpha_1, \qquad d^2-r^2 = a^2,$$

and

$$x = a\sin\beta/(1-\cos\beta).$$

We have therefore on the straight edge

$$\widehat{\beta\beta} = -4P_1\frac{r^2(x^2-d^2+r^2)}{(x^2+d^2-r^2)^2}. \qquad (61)$$

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This has a maximum tension at the symmetrical point ($x = 0$) of magnitude

$$4\mathrm{P}_1 r^2/(d^2-r^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

At the points $x = \pm\sqrt{(d^2-r^2)}$ it vanishes, and then becomes a compression which reaches a maximum value at points at distances $\pm\sqrt{3}\,(d^2-r^2)$ on either side of the foot of the perpendicular from the centre of the hole, which is numerically equal to one-eighth of maximum tension.

The stress round the circular hole may be represented by a simple geometrical construction. If in fig. 3 the centre of the circular hole is C, Q is any point on the circle, and CA the perpendicular drawn from C to the straight edge, and if Q denote the angle QAC, we easily see that

$$\tan\phi = \sin\beta \operatorname{cosech} \alpha_1,$$

and the stress round the circular hole is

$$\widehat{\beta\beta} = \mathrm{P}_1\,(1+2\tan^2\phi). \quad . \quad . \quad . \quad . \quad (63)$$

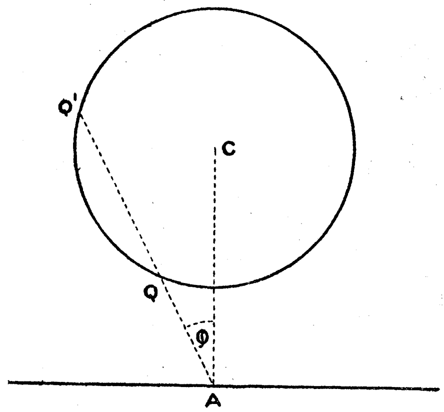

Fig. 3.

Hence the stress is the same at points Q, Q′ which lie on the same ray through A. The stress is minimum at the points nearest to and most remote from the straight edge, where it is a tension P numerically equal to the applied pressure. Thus at these points the stress is the same as it would be in the absence of the straight boundary if the plate were infinite. The maximum stresses are at the points of contact of the tangents drawn from A the circular boundary. At these points its value is

$$\mathrm{P}_1\,\frac{d^2+r^2}{d^2-r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

The maximum tension in the circular boundary is equal to the maximum tension in the straight edge if $d = \sqrt{3}r$. In this case each is equal to $2\mathrm{P}_1$. If the distance of the hole from the straight edge is greater than this value the maximum tension is at a point on the circular boundary; and if it is less, the maximum stress tension is at the symmetrical point on the straight edge. On the other hand, the point of maximum difference of principal stresses is on the straight edge or the circular boundary, according as d is greater or less than $\sqrt{2}r$. This suggests a simple method of determining whether, for a particular material, rupture occurs at the point of greatest tension or at the point of greatest stress-difference. If a circular hole is bored near the straight edge of a uniform plate, so that the distance of its centre from the edge is greater than $\sqrt{2}$ and less than $\sqrt{3}$ of the radius, and a uniform radial pressure is exerted over the hole in any convenient way and increased until

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rupture occurs, the crack will begin on the straight edge according to the greatest tension theory, and on the edge of the hole if the greatest stress-difference theory holds.

It will be noted that the stresses produced will become large if the hole is near to the straight edge. The formulæ are so simple that it is hardly worth tabulating their numerical values, but a single example will serve as an illustration. If the shortest distance from the hole to the straight edge is one-tenth of the radius of the hole, the maximum tension in the straight edge is 19·5 times the pressure in the hole.

§7. A Semi-infinite Plate Containing an Unstressed Circular Hole and Under a Uniform Tension Parallel to its Straight Edge.

Let the circular boundary be defined by $\alpha = \alpha_1$, so that if r is its radius and d the distance of its centre from the straight edge,

$$r = a \operatorname{cosech} \alpha_1, \qquad d = a \coth \alpha_1, \qquad d/r = \cosh \alpha_1.$$

At a distance from the hole the stress-function may be taken as $\chi = \frac{1}{2}\mathrm{T}y^2$ where T is the tension, so that, if $\alpha > 0$,

$$h\chi_0 = \tfrac{1}{2}a\mathrm{T} \sinh^2\alpha/(\cosh \alpha - \cos \beta)$$
$$= \tfrac{1}{2}a\mathrm{T} \sinh \alpha \left(1 + 2 \sum_{n=1}^{\infty} e^{-n\alpha} \cos n\beta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

We have to add to this a stress-function which gives no stress at infinity and no stress over $\alpha = 0$, and is such that the complete stress-function gives no stress over $\alpha = \alpha_1$.

We may omit the term in K in (28), since in this case the region considered lies entirely on one side of $\alpha = 0$, and clearly the required stress-function is even in β. It may readily be seen that the condition that $\widehat{\alpha\alpha}$ and $\widehat{\alpha\beta}$ shall vanish over $\alpha = 0$ is satisfied by (28) if $\phi_n(0) = 0$ and $\phi'_n(0) = 0$ for $n \geqq 1$, and hence from (26) and (27) $\mathrm{A}_n + \mathrm{B}_n = 0$ and $(n+1)\,\mathrm{C}_n + (n+1)\,\mathrm{D}_n = 0$. We may therefore take for our complete stress-function

$$h\chi = a\mathrm{T}\left[\tfrac{1}{2} \sinh \alpha \left\{1 + 2 \sum_{n=1}^{\infty} e^{-n\alpha} \cos n\beta\right\} + \mathrm{B}_0\alpha\,(\cosh \alpha - \cos \beta) + \mathrm{A}_1\,(\cosh 2\alpha - 1) \cos \beta \right.$$
$$\left. + \sum_{n=2}^{\infty} \left\{ \begin{array}{l} \mathrm{A}_n\,[\cosh\,(n+1)\,\alpha - \cosh\,(n-1)\,\alpha] \\ + \mathrm{E}_n\,[(n-1)\sinh\,(n+1)\,\alpha - (n+1)\sinh\,(n-1)\,\alpha] \end{array} \right\} \cos n\beta \right] . \quad . \quad (66)$$

At infinity $\alpha = 0$, $\beta = 0$ the first series diverges, but may of course be replaced by the alternative form in (65). If the second series converges it is clear that at infinity $\chi = \chi_0$.

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We must now choose the coefficients in (66) so as to satisfy (48) and (49), and there is no difficulty in finding the following values for the coefficients :—

$$A_1 = \tfrac{1}{2}e^{-2\alpha_1}\operatorname{sech} 2\alpha_1, \qquad B_0 = \operatorname{sech} 2\alpha_1 \quad . \quad . \quad . \quad (67)$$

$$A_n = -\frac{n^2 \sinh^2\alpha_1 - n \sinh\alpha_1 \cosh\alpha_1 + e^{-n\alpha_1}\sinh n\alpha_1}{2\{\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1\}} \quad . \quad . \quad . \quad (68)$$

$$E_n = \frac{n\sinh^2\alpha_1}{2\{\sinh^2 n\alpha_1 - n^2 \sinh^2\alpha_1\}}.$$

Substituting in (66) we have for the complete stress-function,

$$h\chi = a\mathrm{T}\left[\alpha \operatorname{sech} 2\alpha_1(\cosh\alpha - \cos\beta) + \tfrac{1}{2}\sinh\alpha + \operatorname{sech} 2\alpha_1 \cosh(2\alpha_1 - \alpha)\sinh\alpha\cos\beta + \sum_{n=2}^{\infty} \frac{\left\{\begin{array}{l} n\sinh\alpha_1\sinh(\alpha-\alpha_1)\sinh n\alpha \\ \quad -\sinh\alpha\sinh n(\alpha-\alpha_1)\sinh n\alpha_1\end{array}\right\}}{\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1}\cos n\beta\right]. \quad (69)$$

We may now calculate the stress $\widehat{\beta\beta}$ in the boundaries by means of (6). We find, on the circular boundary $\alpha = \alpha_1$,

$$\widehat{\beta\beta}_1 = 2\mathrm{T}(\cosh\alpha_1 - \cos\beta)\left\{\sinh\alpha_1 \operatorname{sech} 2\alpha_1 + \sum_{n=2}^{\infty} M_n \cos n\beta\right\} \quad . \quad . \quad . \quad (70)$$

where

$$M_n = \frac{n(n-1)\sinh(n+1)\alpha_1 - n(n+1)\sinh(n-1)\alpha_1}{2\{\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1\}} \quad . \quad . \quad . \quad (71)$$

The stress in the straight boundary cannot be directly determined from (69), for it is found that the resulting series diverges for $\alpha = 0$. We can, however, find without difficulty from (66) that when $\alpha = 0$,

$$\widehat{\beta\beta}_0 = \mathrm{T}\left\{1 + (1-\cos\beta)\sum_{1}^{\infty} P_n \cos n\beta\right\} \quad . \quad . \quad . \quad (72)$$

where $P_n = 4nA_n$.

The series in (70) converges only slowly, unless α_1 is large, and for convenience in computation we may transform it by separating the more slowly converging part.

Let

$$M_n = 2n(n\sinh\alpha_1 - \cosh\alpha_1)e^{-n\alpha_1} + N_n, \quad . \quad . \quad . \quad (73)$$

and we readily obtain

$$2(\cosh\alpha_1 - \cos\beta)\sum_{n=1}^{\infty} n(n\sinh\alpha_1 - \cosh\alpha_1)e^{-n\alpha_1}\cos n\beta = 1 - \frac{2\sinh^2\alpha_1\sin^2\beta}{(\cosh\alpha_1 - \cos\beta)^2},$$

Substituting in (70) we have

$$\widehat{\beta\beta}_1 = 2\mathrm{T}\left\{1 - \frac{2\sinh^2\alpha_1\sin^2\beta}{(\cosh\alpha_1-\cos\beta)^2}\right\} + 2\mathrm{T}(\cosh\alpha_1 - \cos\beta)\left\{\sinh\alpha_1\operatorname{sech}2\alpha_1 + 2e^{-2\alpha_1}\cos\beta + \sum_{n=2}^{\infty} N_n\cos n\beta\right\}. \quad (74)$$

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If θ is the angle between the radius to the point α_1, β and the perpendicular to the straight edge, then

$$\sin\theta = \frac{\sinh\alpha_1 \sin\beta}{\cosh\alpha_1 - \cos\beta},$$

and if α_1 is large (74) reduces to $\widehat{\beta\beta_1} = \mathrm{T}(1+2\cos 2\theta)$, which agrees with the known result for a hole in an infinite plate and gives compression numerically equal to T at the extremities of the diameter parallel to the tension, and tensions equal to 3T at the extremities of the perpendicular diameter.

The numerical values of the coefficients P_n, N_n are given in Tables I. and II. respectively. It will be noted from Table I. (that, as α_1 increases, P_2 tends to become

TABLE I.

α_1.	0·6.	0·8.	1·0.	1·2.	1·4.	1·6.	1·8.	2·0.	2·2.	2·4.
P_1	0·3327	0·1567	0·0719	0·0327	0·0147	0·0066	0·0030	0·0013	0·0006	0·0003
$-\mathrm{P}_2$	3·5861	2·0401	1·2545	0·7987	0·5180	0·3400	0·2247	0·1493	0·0994	0·0664
$-\mathrm{P}_3$	2·2393	1·0622	0·5110	0·2448	0·1160	0·0543	0·0251	0·0115	0·0053	0·0024
$-\mathrm{P}_4$	1·3557	0·4874	0·1699	0·0570	0·0185	0·0059	0·0018	0·0006	0·0002	0·0001
$-\mathrm{P}_5$	0·7602	0·1970	0·0474	0·0108	0·0024	0·0005	0·0001			
$-\mathrm{P}_6$	0·3964	0·0713	0·0116	0·0018	0·0003					
$-\mathrm{P}_7$	0·1934	0·0237	0·0026	0·0003						
$-\mathrm{P}_8$	0·0891	0·0073	3·0005							
$-\mathrm{P}_9$	0·0391	0·0022	0·0001							
$-\mathrm{P}_{10}$	0·0165	0·0005								
$-\mathrm{P}_{11}$	0·0067	0·0002								
$-\mathrm{P}_{12}$	0·0027									
$-\mathrm{P}_{13}$	0·0010									
$-\mathrm{P}_{14}$	0·0004									
$-\mathrm{P}_{15}$	0·0001									
$-\mathrm{P}_{16}$	0·0001									

TABLE II.

α_1.	0·6.	0·8.	1·0.	1·2.	1·4.	1·6.	1·8.	2·0.	2·2.	2·4.
N_2	1·4649	0·7716	0·4139	0·2240	0·1219	0·0665	0·0364	0·0199	0·0109	0·0060
N_3	0·7457	0·2647	0·0914	0·0306	0·0100	0·0032	0·0010	0·0003	0·0001	
N_4	0·3238	0·0719	0·0148	0·0029	0·0005	0·0001				
N_5	0·1232	0·0162	0·0019	0·0002						
N_6	0·0421	0·0032	0·0002							
N_7	0·0131	0·0005								
N_8	0·0038	0·0001								
N_9	0·0010									
N_{10}	0·0003									
N_{11}	0·0001									

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large compared to the other coefficients. Hence, when the hole is at a considerable distance from the straight edge, the stress in the straight edge approximates to

$$T\{1-C\cos 2\beta(1-\cos\beta)\}$$

where C is a small positive constant.

This shows that the stress in the straight edge is a minimum at the mid-point, increases to a maximum as we move outwards, then diminishes to a second minimum, and finally increases steadily to the value T at infinity, where $\beta = 0$.

In fig. 4 we have plotted the graphs of the stresses in the boundaries for a case in

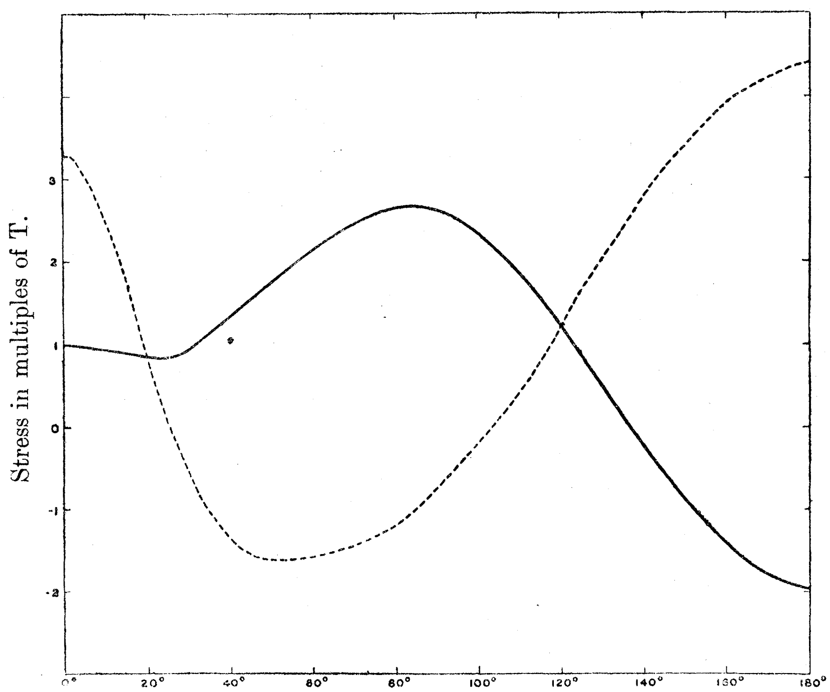

Fig. 4.

which the hole is fairly near to the straight edge, $\alpha_1 = 0{\cdot}8$, for which the shortest distance between the two boundaries is approximately one-third of the radius of the circle. It will be noted that the general character of the stresses is not affected by the proximity of the straight edge. It will be remembered that when the hole is at a great distance from the straight edge there are maximum stresses of 3T at the extremities of the diameter perpendicular to the straight edge, with points of maximum compression numerically equal to T lying between. For $\alpha_1 = 0{\cdot}8$ we find that the maxima occur at the same places but are increased, the increase being more marked at the point nearest to the straight edge, where the tension is 4·366T, while its value at the point most remote from the straight edge is 3·266T. The stress in the straight edge also maintains the same general character as it exhibits when the hole is at a great distance

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from the straight edge. Here again, however, the maxima and minima are accentuated. The minimum at the central point has decreased and has become a compression numerically equal to 1·956T.

It appears that for the range of values which we have investigated the maximum stress is on the circular boundary at the point of nearest approach to the straight edge. Its value for different values of the ratio of the distance of the centre of the hole from the straight edge to the radius of the hole, together with the stresses at the centre of the straight edge and at the most remote point of the circular boundary, is shown in Table III.

TABLE III.

α_1.	Ratio of distance of centre from edge to radius of hole.	Stress at mid-point of straight edge.	Stress at nearest point of circular boundary.	Stress at most remote point of circular boundary.
0·6	1·185	−4·080T	5·064T	3·362T
0·8	1·337	−1·956	4·366	3·266
1·0	1·543	−0·895	3·919	3·201
1·2	1·811	−0·269	3·609	3·152
1·4	2·151	+0·134	3·396	3·115
1·6	2·577	0·405	3·254	3·087
1·8	3·107	0·591	3·162	3·065
2·0	3·762	0·721	3·103	3·048
2·2	4·568	0·810	3·065	3·035
2·4	5·557	0·871	3·043	3·025
∞	∞	1·000	3·000	3·000

It will be noted that when the hole is very near to the straight edge, so that the two boundaries are separated only by a narrow connecting piece, the stress in this piece consists of a very large tension on the inside and a numerically slightly less compression on the outside. Hence, as might be expected from general considerations, the stress in this narrow connecting piece is a bending moment accompanied by a certain amount of tension.

These results may be compared with some experimental results recently obtained by Prof. COKER and Messrs. K. C. CHAKKO and Y. SATAKE* by optical means. These deal with the stresses in a strip of *finite* width under tension with a circular hole centrally placed, whereas we have considered the case of a semi-infinite plate with a circular hole near its straight edge. The problems are therefore not quite comparable; but as in each case the critical region will clearly be near the minimum section between the hole and a straight boundary, the two problems may be expected to exhibit the same general characteristics. For the strip of finite width it is found that there is maximum stress in the circular boundary at the points of nearest approach to the

* 'Transactions of the Institution of Engineers and Shipbuilders in Scotland,' vol. lxiii., Part I., p. 33, 1919.

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straight edges and minimum stress at the points of the straight edge immediately opposite the centre of the hole. Moreover as the radius of the hole is increased in proportion to the distance of its centre from the edges of the strip these maxima and minima become more pronounced. In all the cases examined experimentally the minimum stress in the straight edge remains a tension, but Prof. COKER surmises that if the radius of the hole were still further increased in proportion to the width of the strip this minimum stress would become a compression. All these results agree qualitatively with the theoretical results established in this paper for the semi-infinite plate, and allowing for the difference in the two problems they may be taken as a substantial experimental verification.

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